

# Persistence of a Brownian particle in a Time Dependent Potential

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We investigate the persistence probability of a Brownian particle in a harmonic potential, which decays to zero at long times – leading to an unbounded motion of the Brownian particle. We consider two functional forms for the decay of the confinement, an exponential and an algebraic decay. Analytical calculations and numerical simulations show, that for the case of the exponential relaxation, the dynamics of Brownian particle at short and long times are independent of the parameters of the relaxation. On the contrary, for the algebraic decay of the confinement, the dynamics at long times is determined by the exponent of the decay. Finally, using the two-time correlation function for the position of the Brownian particle, we construct the persistence probability for the Brownian walker in such a scenario.

## I. INTRODUCTION

The phenomenon of persistence has been of continuing interest in the past decade. Persistence is quantified through the persistence probability  $p(t)$ , that a stochastic variable has not changed its sign over a time  $t$ . In a wide class of non-equilibrium systems this probability decays algebraically with an exponent  $\theta$  and the exponent has been studied in systems that include free random walk in homogeneous [1, 2] and disordered media [3], critical dynamics [4], surface growth [5–11], polymer dynamics [12], diffusive processes with random initial condition [13–15], advected diffusive process [16] and finance [17, 18]. A precise theoretical prediction for  $p(t)$  can only be worked out only for a select few cases [19] – the simplest scenario being an exponentially decaying stationary correlator, as in the case of an overdamped Brownian motion. In general, for most Gaussian stochastic processes the decay of the stationary correlator,  $C(T) \equiv \langle \bar{X}(T)\bar{X}(0) \rangle$ , is non-exponential. The behavior of  $C(T)$  in the neighborhood of zero characterizes the density of zero crossings for the underlying stochastic process [13, 19]. When  $C(T)$  near zero, has a quadratic dependence on time in the first order, the number of zero crossings of the stochastic process is finite, and the exponent  $\theta$  is extracted using the Independent Interval Approximation (IIA) [13] or the sign time distribution of the stochastic variable [15]. Conversely, when  $C(T) \sim 1 - \mathcal{O}(T^\alpha)$ , with  $\alpha < 2$ , the density of zero crossings is infinite and perturbation expansions about a random walk correlator gives a good estimate of the persistence exponent [6].

The simplest of all these systems, which exhibit an algebraic decay of  $p(t)$  with an exponent  $1/2$ , is the case of a overdamped Brownian particle. Lying in the interface of science and engineering, Brownian motion is ubiquitous around us and plays a dominant role in the nano and mesoscopic world. The underlying principle

of this stochastic process is not only used for theoretical modeling of a wide range of complex phenomena [20], but Brownian motion in itself serves as an experimental tool for probing microscopic environments [20–23]. In the popular Langevin picture, the erratic motion of a Brownian particle is well described by Newton’s equation of motion with a viscous drag and a delta-correlated stochastic force acting on the particle. While the non-Markovian nature of the phenomenon can be taken into consideration by using a generalized Langevin equation with a finite correlation time for the stochastic noise and a memory dependent friction, in the following discussion we shall restrict ourselves to the Markovian scenario.

In this article, we investigate the persistence probability of a Brownian particle in a time dependent potential – a scenario corresponding to the trapping of a tracer particle in some potential which eventually relaxes to zero. To keep the following discussions at an analytically tractable level, we choose a harmonic potential, given by  $U(x, t) = \frac{1}{2}f(t)x^2$ . The function  $f(t)$  can be viewed as time-dependent spring constant, with  $f(t) \rightarrow 0$  as  $t \rightarrow \infty$ , so that the particle motion becomes unbounded in the long-time limit. The converse situation of a constant confinement strength has already been studied in Ref. [19, 24, 25]. In the Fokker-Planck description, the calculation of the persistence probability translates to solving the backward Fokker-Planck equation, with an absorbing wall at the  $x = 0$ . An alternative approach to determine the survival probability, as outlined in [1, 19, 24], is from the two-time correlation function for the position of the stochastic variable  $x$  – exploiting the fact that for a Gaussian stationary process with a correlator decaying exponentially at all times, the persistence probability also decays exponentially.

The rest of the article is organized as follows: we introduce the dynamical equations of motion and construct the two-time correlation functions in section II. A discussion on the mean-square displacement of the Brownian walker and the relevant time scales due to the time dependent trap is also presented in this section. We study two types of relaxation phenomena – an exponential and an algebraic relaxation of the confinement discussed in section II A and section II B, respectively. Finally, the

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persistence probability is discussed in section III.

## II. BROWNIAN PARTICLE IN A TIME-DEPENDENT POTENTIAL

For simplicity, we take the overdamped limit for which the dynamics of a Brownian particle, with a unit mass, is governed by

$$\dot{x} = -f(t)x(t) + \eta(t), \quad (1)$$

where  $\eta(t)$  is the stochastic velocity characterizing the solvent. The above equation is further supplemented by the moments of the stochastic noise,

$$\langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t)\eta(t') \rangle = 2D\delta(t-t'). \quad (2)$$

At this point, we assume that the stochastic noise in “internal”, characterized by the viscosity and the temperature of the solvent, while the time-dependent confinement is “external” and does not change the delta correlation. An experimental realization of the model system would correspond to a laser trapping of a tracer, with the intensity of laser decaying in time. In such a scenario, the transport parameters and the temperature  $T$  gets renormalized [26, 27]. The corresponding solution to (1) is given by

$$x(t) = e^{-\int_0^t f(t')dt'} \int_0^t dt_1 \eta(t_1) e^{\int_0^{t_1} f(t')dt'_1}, \quad (3)$$

with the initial condition  $x(0) = 0$ . Denoting  $g(t) = \int_0^t f(t')dt'$ , the two time correlation function can be constructed from (3),

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle &= e^{-g(t_1)}e^{-g(t_2)} \int_0^{t_1} dt'_1 \int_0^{t_2} dt'_2 \\ &\quad \langle \eta(t'_1)\eta(t'_2) \rangle e^{g(t'_1)}e^{g(t'_2)}. \end{aligned} \quad (4)$$

### A. Exponential relaxation

We first consider the case when the relaxation of the potential is given by an exponential decay,  $f(t) = \lambda e^{-t/\tau}$ . There are two time-scales in the system,  $\tau$  and  $\lambda^{-1}$ . The later determines the time scale when the Brownian particle is confined in the potential, whereas the former determines the relaxation of the potential (fig. 1a). We further consider the situation when  $\tau > \lambda^{-1}$ ; the relaxation time-scale of the potential is larger than the entrapment time-scale  $\lambda^{-1}$ . When  $\tau < \lambda^{-1}$ , the confinement decays even before the particle can be trapped, with the result that the Brownian particle undergoes free diffusion.

Consequently, the function  $g(t)$  takes the form  $g(t) = \lambda\tau(1 - e^{-t/\tau})$ . Substituting for  $g(t)$  in (4) and subsequently using (2) we arrive at

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle &= 2De^{\lambda\tau e^{-t_1/\tau}} e^{\lambda\tau e^{-t_2/\tau}} \\ &\quad \int_0^{t_2} dt'_2 e^{-2\lambda\tau e^{-t'_2/\tau}}, \end{aligned} \quad (5)$$

with the assumption that  $t_1 > t_2$ . Performing the integral over  $t'_2$  in (5), the two-time correlation function becomes,

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle &= 2D\tau e^{\lambda\tau e^{-t_1/\tau}} e^{\lambda\tau e^{-t_2/\tau}} [\text{Ei}(-2\lambda\tau) \\ &\quad - \text{Ei}(-2\lambda\tau e^{-t_2/\tau})], \end{aligned} \quad (6)$$

where  $\text{Ei}(x)$  is the exponential integral defined as

$$\text{Ei}(t) = - \int_{-t}^{\infty} z^{-1} e^{-z} dz. \quad (7)$$

Using (6), the mean-square displacement  $\langle x^2(t) \rangle$  reads as,

$$\langle x^2(t) \rangle = 2D\tau e^{2\lambda\tau e^{-t/\tau}} [\text{Ei}(-2\lambda\tau) - \text{Ei}(-2\lambda\tau e^{-t/\tau})]. \quad (8)$$

At this point, it is instructive to construct the limiting behaviors of the mean-square displacement – when  $t < \lambda^{-1}$  and  $t > \tau$ . There are two scenarios we consider below – the first, when  $\tau$  is large so that the limit  $\tau \rightarrow \infty$  is appropriate, and the second when  $\tau$  remains finite. In the limit of  $\tau \rightarrow \infty$ , the relaxation of the potential is slow and the Brownian particle feels a constant confinement strength  $\lambda$  and (8) reduces to

$$\langle x^2(t) \rangle = \frac{D}{\lambda} (1 - e^{-2t\lambda}) + \mathcal{O}(\tau^{-1}). \quad (9)$$

To construct the corresponding two-time correlation function, we expand the exponentials in (5) and keep the terms which are independent of  $\tau$ . The evaluation of the integral over  $t'_2$  then gives,

$$\langle x(t_1)x(t_2) \rangle = \frac{D}{\lambda} [e^{-\lambda(t_1-t_2)} - e^{-\lambda(t_1+t_2)}], \quad (10)$$

which is exactly the correlation function for a non-stationary Ornstein-Uhlenbeck process [25]. Eventually, for  $t \gg \tau$  the particle motion becomes unbounded and the mean-square displacement grows linearly with time. A formal quantitative result in this limit can be derived if we take the limit of  $t \rightarrow \infty$  in (8) and expand the term within the brackets to get

$$\begin{aligned} \langle x^2(t) \rangle &= 2D\tau e^{2\lambda\tau e^{-t/\tau}} [-\gamma + 2\lambda\tau e^{-t/\tau} - \lambda^2\tau^2 e^{-2t/\tau} \\ &\quad + \text{Ei}(-2\lambda\tau) + \frac{t}{\tau} + \frac{1}{2} \ln(1/2\lambda^2\tau)], \end{aligned} \quad (11)$$

where  $\gamma$  is the Euler’s constant with a numerical value of  $\sim 0.5772$ . Keeping in mind that  $t \gg \tau$ , the exponential functions in (11) can be ignored in comparison to the linearly growing term which survives, so that we recover the classic diffusion of the Brownian particle with  $\langle x^2(t) \rangle = 2Dt$ . In the opposite limit of  $t \rightarrow 0$ , a Taylor expansion of (8) yields,

$$\langle x^2(t) \rangle = 2Dt + \mathcal{O}(t^2). \quad (12)$$

The two limiting behaviors in (11) and (12) are completely independent of the time scales, and therefore, do not contain any information about the confinement potential. On the contrary, when the relaxation is slow ( $\tau \rightarrow \infty$ ), only the short-time dynamics is independent of  $\lambda$  or  $\tau$ . The asymptotic mean-square displacement, in the limit of a slow relaxation, is constant in time and is determined by the ratio  $D/\lambda$ .

### B. Algebraic relaxation

We now consider our second choice for the relaxation dynamics of the harmonic potential – an algebraic decay of the time dependent spring constant,

$$U(x, t) = \frac{\lambda}{2} \left( \frac{\tau}{t} \right)^\alpha x^2, \quad (13)$$

with  $\alpha \leq 1$ . Using (4), the two-time correlation function is

$$\langle x(t_1)x(t_2) \rangle = \frac{2D}{1-\alpha} e^{-\lambda_1 t_1^{1-\alpha}} e^{-\lambda_1 t_2^{1-\alpha}} \int_0^{t_2^{1-\alpha}} du u^{\alpha/(1-\alpha)} e^{2\lambda_1 u} \quad (14)$$

where  $\lambda_1 = \lambda\tau^\alpha/(1-\alpha)$ . The integral (14) over  $u$  yields

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle &= \frac{2D}{1-\alpha} e^{-\lambda_1 t_1^{1-\alpha}} e^{-\lambda_1 t_2^{1-\alpha}} \\ &\quad (-2\lambda_1)^{-1/(1-\alpha)} \gamma \left( \frac{1}{1-\alpha}, -2\lambda_1 t_2^{1-\alpha} \right) \end{aligned} \quad (15)$$

where the integral  $\gamma$  is the lower incomplete gamma function defined as

$$\gamma(a, z) = \int_0^z e^{-u} u^{a-1} du \quad (16)$$

and  $\Gamma(a, z)$  is the upper incomplete Gamma function satisfying  $\Gamma(a, z) + \gamma(a, z) = \Gamma(a)$ . The numerical value of  $\gamma(a, z)$  can be evaluated using Gauss's continued fraction, which converges for all values of  $z$ .

Substituting  $t_1 = t_2 = t$ , the mean-square displacement is given by

$$\begin{aligned} \langle x^2(t) \rangle &= \frac{2D}{1-\alpha} e^{-2\lambda_1 t^{1-\alpha}} (-2\lambda_1)^{-1/(1-\alpha)} \\ &\quad \gamma \left( \frac{1}{1-\alpha}, -2\lambda_1 t^{1-\alpha} \right) \end{aligned} \quad (17)$$

Unlike the exponential relaxation of the potential, there is a single crossover time scale which emerges from (17)

$$\bar{\tau} = \left( \frac{1-\alpha}{\lambda\tau^\alpha} \right)^{1/(1-\alpha)}, \quad (18)$$

and separates the regimes of normal diffusion and sub-diffusion in the system (fig. 1b). A Taylor expansion of (17) for  $t < \bar{\tau}$  gives,

$$\langle x^2(t) \rangle = 2Dt + \mathcal{O}(t^{2-\alpha}) \quad \text{for } t < \bar{\tau}, \quad (19)$$

while the asymptotic expansion yields

$$\langle x^2(t) \rangle = \left( \frac{D}{\lambda\tau^\alpha} \right) t^\alpha + \mathcal{O}(t^{-(1-2\alpha)}) \quad \text{for } t > \bar{\tau}. \quad (20)$$

This counter intuitive result can be understood by considering the motion of a free Brownian particle. In the absence of the confinement potential the Brownian particle moves a distance  $\sqrt{t}$  in time  $t$ . If we now switch on the potential, the strength of the potential becomes  $\lambda(\tau/t)^\alpha t$  and for  $\alpha < 1$  we see that the particle feels the “soft” walls all the time. Mathematically, this argument translates to the fact that the new time scale  $\bar{\tau}$  in (18) diverges as  $\alpha \rightarrow 1$  and is not defined in the real line for  $\alpha > 1$ . We note that for  $\alpha = 0$ , eq. (17) reduces to that of the Ornstein-Uhlenbeck process,

$$\langle x^2(t) \rangle = \frac{D}{\lambda} [1 - e^{-2\lambda t}]. \quad (21)$$

In fig. 1, we show the mean-square displacement of a Brownian particle whose dynamics is governed by (1) and (2), with  $f(t)$  given by an exponential (fig. 1a) and an algebraic (fig. 1b) relaxation. The numerical integration of (1) was done using the Euler scheme with an integration time step of  $dt = 0.001$ . In the numerical solutions, the value of the diffusion coefficient  $D$  was taken as unity. For the exponential relaxation of the confinement, the measured mean-square displacement show three distinct regimes – two diffusive regimes with a crossover in between. For very short ( $t < \lambda^{-1}$ ) and long times ( $t > \tau$ ), the particle does not feel the trap and its motion is purely diffusive, corresponding to (11) – (12). In the intermediate times, we observe a plateau for  $\lambda^{-1} < t < \tau$ , corresponding to the trapping of the particle in the potential. To understand the origin of this plateau, we expand the exponential in (1), and retaining the zeroth order term then leads to a constant confinement, so that the mean-square displacement saturates to a value  $\propto \lambda^{-1}$ . This behavior can be observed in the left panel of fig. 1, the main figure of which presents data for constant  $\lambda$  but different  $\tau$ , while the inset shows data for a constant  $\tau$  but different  $\lambda$ . A comparison shows that the plateau is determined by  $\lambda^{-1}$ . On the contrary, for the algebraic relaxation, since a single time scale emerges from the dynamics, we observe only one crossover regime determined by  $\bar{\tau}$  which separates the diffusive and the sub-diffusive regimes (the right panel of fig. 1).

### III. PERSISTENCE PROBABILITY

To obtain the persistence probability, we take the route outlined in Ref. [1, 19] – we map the non-stationary

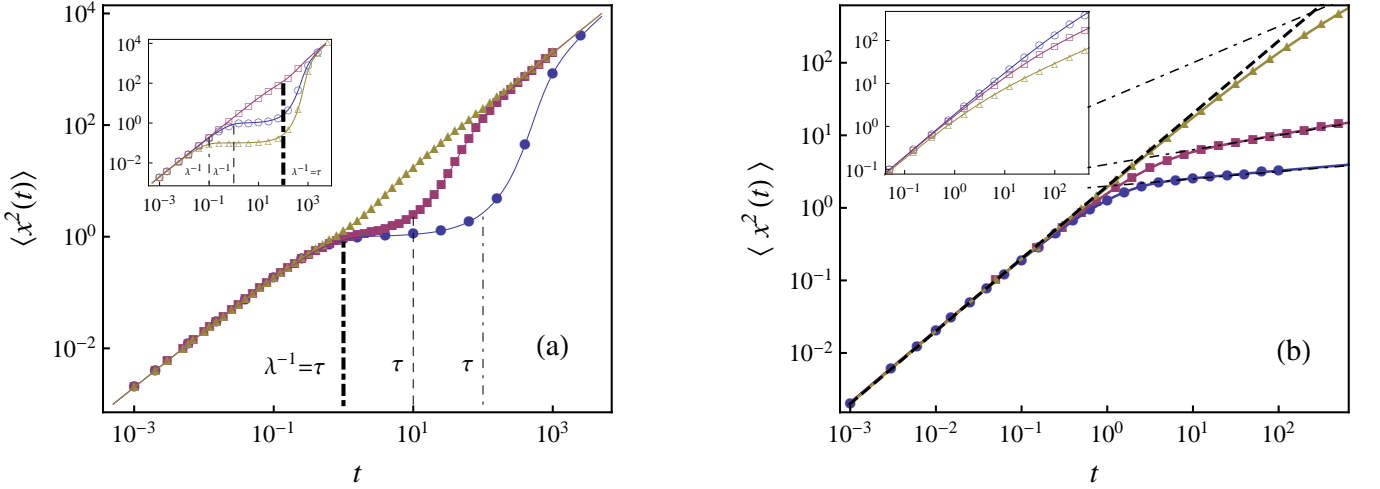


FIG. 1. (Color online) **(a)**: Double logarithmic plot of the mean-square displacement  $\langle x^2(t) \rangle$  for the exponential relaxation of the confinement; with  $\lambda = 1$  and  $\tau = 1.0$  ( $\blacktriangle$ ),  $10.0$  ( $\blacksquare$ ) and  $100.0$  ( $\bullet$ ). The solid lines are the plots of (8) for the corresponding values of  $\tau$  and  $\lambda$ . The thick dot-dashed line in the main plot corresponds to  $\lambda^{-1} = \tau = 1$ , while the dashed and the dot-dashed line corresponds to  $\tau = 10$  and  $\tau = 100$ , respectively. **Inset**: We depict the variation of the mean-square displacement for different values of  $\lambda$  with  $\tau = 100$  fixed;  $\lambda = 0.01$  ( $\square$ ),  $1.0$  ( $\circ$ ) and  $10.0$  ( $\triangle$ ). The solid lines are plot of (8) for the corresponding values of  $\lambda$  and  $\tau$ . The thick dot-dashed line corresponds to  $\lambda^{-1} = \tau = 100$  while the dashed lines denote the values of  $\lambda^{-1} = 0.1$  and  $\lambda^{-1} = 1$ . **(b)**: Plot of mean-square displacement for  $\lambda = 1.0$ ,  $\tau = 0.001$  and  $\alpha = 0.1$  ( $\bullet$ ),  $0.2$  ( $\blacksquare$ ) and  $0.5$  ( $\blacktriangle$ ) and the corresponding plots of (17) for the three values of  $\alpha$ . The thick dashed black line is the plot of  $2Dt$  and the thin dot-dashed black lines are the plots of (20) for the corresponding values of  $\alpha$ . **Inset**: Plot of mean-square displacement for  $\lambda = 1.0$ ,  $\alpha = 0.5$  but with  $\tau = 0.001$  ( $\circ$ ),  $0.01$  ( $\square$ ) and  $0.1$  ( $\triangle$ ).

process  $x(t)$  to a stationary Ornstein-Uhlenbeck process. This is usually achieved, first by a normalization of  $x(t)$  by  $\sqrt{\langle x^2(t) \rangle}$ , the root-mean-square distance the particle has traveled and then using a suitable transformation in time. Once, we have the stationary process  $\bar{X}$ , with correlator  $C(T)$ , the persistence problem reduces to calculation of no zero crossing of  $\bar{X}$ . When  $C(T)$  is a purely exponential decay for all times, the persistence probability is the solution to the backward Fokker-Planck equation for an Ornstein-Uhlenbeck process, which can be shown to decay as  $P(T) = \frac{2}{\pi} \sin^{-1}[C(T)]$ , [1, 24]. An application of this method, therefore, requires the transformation of the stochastic process  $x(t)$  to a Gaussian stationary process. Since the correlation function in (6) and (15) is non-stationary, we make the following transformations – we first define the normalized variable  $\bar{X}(t) \equiv \frac{x(t)}{\sqrt{\langle x^2(t) \rangle}}$  and construct the two-time correlation function, following which we make a suitable transformation in time to make the correlator stationary, as well as an exponentially decaying function for all times.

Before we proceed to give a derivation of the persistence probability for the two models introduced above, we derive a general result applicable to the model system in (1). To transform the nonstationary process in (1), we consider the transformations  $\bar{X} = x(t)/l(t)$  and  $e^T = l^2(t)e^{2g(t)}$ , where  $l^2(t) = \langle x^2(t) \rangle$  and  $g(t) = \int f(t')dt'$ . Substituting these transformations in (1), we obtain a

stationary Ornstein-Uhlenbeck process,

$$\frac{d\bar{X}}{dT} = -\frac{1}{2}\bar{X} + \bar{\eta}(T), \quad (22)$$

where  $\bar{\eta}(T)$  is a Gaussian white noise with zero mean and unit variance. The relation between  $\bar{\eta}$  and  $\eta$  can be determined from the transformation of the delta function and takes the form  $\bar{\eta}(T) = \frac{l(t)}{2D_0}\eta(t)$ . The stationary correlator for the process in (22) is then given by  $C(T) = e^{-T/2}$ . Accordingly, the persistence probability in real time decays as  $p(t) \sim e^{-g(t)}/l(t)$ . In the following, we illustrate this explicitly for the two specific cases presented in section II A and section II B.

### A. Exponential Relaxation

The two-time correlation function for  $\bar{X}$  reads,

$$\langle \bar{X}(t_1)\bar{X}(t_2) \rangle = \frac{\langle x(t_1)x(t_2) \rangle}{\sqrt{\langle x^2(t_1) \rangle \langle x^2(t_2) \rangle}} \quad (23)$$

Using (6) and (8) in the above equation, we have

$$\langle \bar{X}(t_1)\bar{X}(t_2) \rangle = \sqrt{\frac{\text{Ei}(-2\lambda\tau) - \text{Ei}(-2\lambda\tau e^{-t_2/\tau})}{\text{Ei}(-2\lambda\tau) - \text{Ei}(-2\lambda\tau e^{-t_1/\tau})}}. \quad (24)$$

The time transformation  $e^T = l^2(t)e^{2g(t)}$  reads,

$$e^T = \text{Ei}(-2\lambda\tau) - \text{Ei}(-2\lambda\tau e^{-t/\tau}), \quad (25)$$

which transforms (24) to

$$\langle \bar{X}(T_1) \bar{X}(T_2) \rangle = e^{-\frac{1}{2}(T_1 - T_2)} \quad (26)$$

The correlator for the stochastic process  $\bar{X}$  is stationary and exponentially decaying. The asymptotic behavior of the persistence probability for such a process is then given by  $P(T) \sim e^{-T/2}$  [19]. Transforming back to real time, the persistence probability for the process  $x(t)$  is then given by,

$$p(t) \sim \left[ \text{Ei}(-2\lambda\tau) - \text{Ei}(-2\lambda\tau e^{-t/\tau}) \right]^{-1/2} \quad (27)$$

For  $t \ll \lambda$  and  $t \gg \tau$ , a Taylor and an asymptotic expansion of the above equation gives  $p(t) \sim t^{-1/2}$ . Finally, in the limit of  $\tau \rightarrow \infty$ , the persistence probability reads as

$$p(t) \sim \left[ (1 - e^{-2\lambda t}) e^{2\lambda t} + \mathcal{O}(\tau^{-2}) \right]^{-1/2} \quad (28)$$

which is identical to the result of Ref.[25]. To determine the persistence probability of the Brownian particle using a numerical integration, we chose an ensemble of random initial conditions in the neighborhood of zero (so that the sign of  $x(0)$  is well defined) and followed the sign change of the position. The fraction of particles which did not change the sign of the coordinates in time  $t$  gives an estimate of the persistence probability. The results presented in (27) (the colored lines in fig. 2) and (28) (the black dashed line in fig. 2) is compared with the measured persistence probability using the numerical simulation of (1) (the solid points) in fig. 2. For short and long times, the persistence probability  $p(t) \sim t^{-1/2}$ , a signature of purely diffusive motion presented in (11) and (12).

### B. Algebraic relaxation

To determine the survival probability, we proceed in the similar way and construct the two-time correlation function for the normalized variable  $\bar{X}$ .

$$\langle \bar{X}(t_1) \bar{X}(t_2) \rangle = \sqrt{\frac{h(t_2)}{h(t_1)}} \quad (29)$$

where the function  $h(t)$  is the bracketed term in (17),

$$h(t) = (-2\lambda_1)^{1/(1-\alpha)} \gamma(1/(1-\alpha), -2\lambda_1 t^{1-\alpha})$$

Defining the time transformation  $e^T \equiv l^2(t) e^{2g(t)} = h(t)$ , the non-stationary correlator in (15) is transformed into a Gaussian stationary correlator which decays exponentially. Following [19], the persistence probability, in real time decays as

$$p(t) \sim \left[ (-2\lambda_1)^{-1/(1-\alpha)} \gamma(1/(1-\alpha), -2\lambda_1 t^{1-\alpha}) \right]^{-1/2} \quad (30)$$

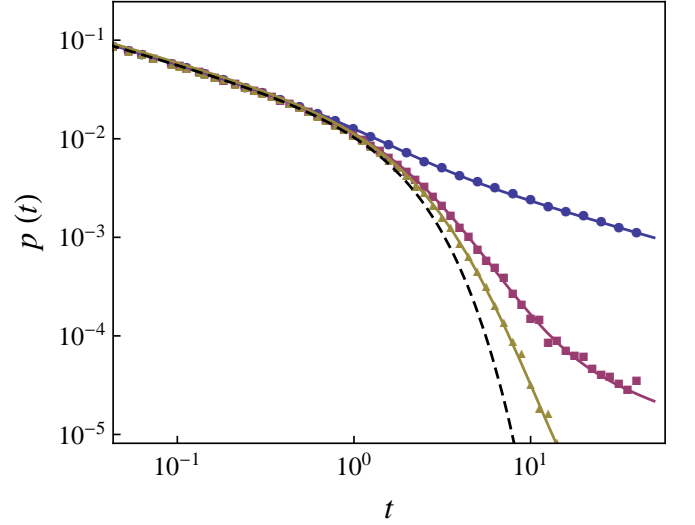


FIG. 2. (Color online) Double logarithmic plot of persistence probability where  $f(t)$  decays exponentially, for  $\lambda = 1.0$  and  $\tau = 1.0$  (●),  $5.0$  (■) and  $10.0$  (▲). The solid and the dashed lines are the plots of (27) for the corresponding values of  $\tau$  and dashed line is the plot of (28).

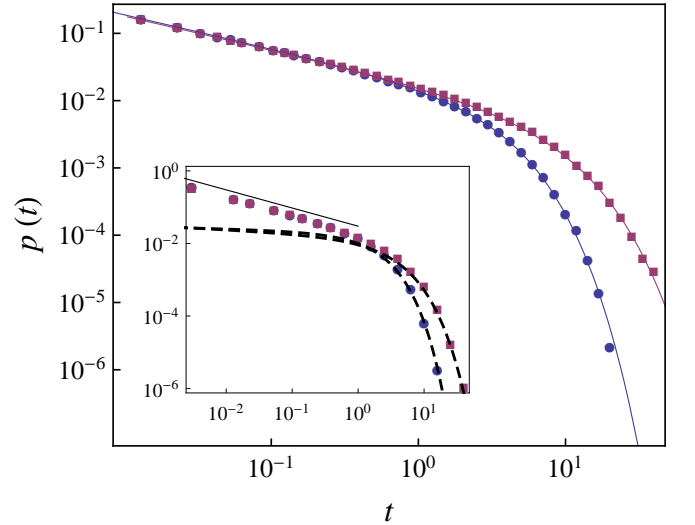


FIG. 3. (Color online) Plot of persistence probability for  $\lambda = 1.0$ ,  $\tau = 0.001$   $\alpha = 0.1$  (●) and  $0.2$  (■). The solid lines in the main plot are plots of (30) for the corresponding values of  $\alpha$ . **Inset:** Plot of the persistence probability for  $\lambda = 1$ ,  $\tau = 0.01$  with  $\alpha = 0.1$  (●) and  $0.2$  (■). The solid black line is the plot of  $t^{-1/2}$  and the dashed lines are the plots of the asymptotic expansion (32).

We next consider the limiting behaviors of the persistence probability given in (30). Substituting  $\alpha = 0$  in (30), the probability reduces to the case of a harmonically confined Brownian particle with constant confinement strength [25],

$$p(t) \sim [e^{2\lambda t} (1 - e^{-2\lambda t})]^{-1/2} \quad (31)$$

For a finite value of  $\alpha < 1$ , when  $t < \bar{\tau}$ , the persistence probability decays as  $p(t) \sim t^{-1/2}$  while an asymptotic expansion of (30) gives,

$$p(t) \sim \frac{1}{t^{\alpha/2}} e^{-(t/\tau)^{1-\alpha}}. \quad (32)$$

In fig. 3, we compare the results of (30), (31) and (32) with the measured persistence probability from the numerical integration of (1). The colored lines in the figure correspond to Eq. (30), while the dot-dashed lines are plots of Eq. (32). At short times, the motion is purely diffusive, and therefore we observe a  $t^{-1/2}$  decay of  $p(t)$  (the solid line in fig. 3).

We note, that even though the mean-square displacement for  $t \gg \tau$  is similar to that of fractional Brownian motion, the decay of the persistence probabilities in the two scenarios are entirely different. For a particle which performs a fractional Brownian motion, the corresponding steady state persistence probability decays purely algebraically with an exponent  $1 - \alpha/2$  [6].

### C. Effect of Inertia

Finally, before concluding, we remark upon the divergence of  $p(t)$  as  $t \rightarrow 0$ . This singularity is entirely the artifact of coarse-graining in (1), where we have neglected the inertia term. Strictly speaking, at this level of coarse-graining, we are not allowed to take the  $t \rightarrow 0$  limit, since the inertia of the particle plays a important role at such short times. The inclusion of the inertia term changes the short-time dynamics of the particle at to a deterministic one, as opposed to purely diffusive motion observed in the overdamped limit. Since the motion is now deterministic, the particle is persistently driven away from its initial position (the velocities remain strongly correlated), with the effect that the survival probability becomes constant. The purpose of this section is not only to demonstrate this fact that the inertia term in the Langevin equation indeed removes the singularity in the persistence probability, but is also motivated by the recent experimental evidence of the ballistic regime of a Brownian particle [28, 29]. While an accurate analysis would correspond to solving (1) with the inertia term included, but it becomes difficult to extract any information from the resulting expressions. However, since we are looking at a time much shorter than  $\lambda^{-1}$ , exclusion of the confinement is justified – as the particle does not feel the confinement at such small times.

To this end, we consider the complete Langevin equation for the momentum of a particle without any potential confinement,

$$\dot{p} = -\frac{\gamma}{m}p + \eta \quad (33)$$

together with  $\langle \eta(t) \rangle = 0$  and  $\langle \eta(t)\eta(t') \rangle = 2k_B T \gamma \delta(t - t')$ . Since at short times, the dynamics of a Brownian particle in the model systems presented above is purely

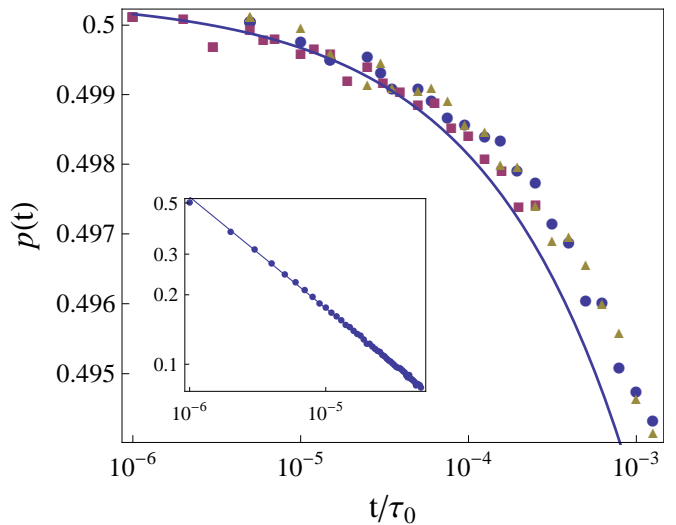


FIG. 4. (Color online) The main figure depicts a double logarithmic plot of persistence probability of a Brownian walker with the inertia term included in the Langevin equation for the ratio of  $\gamma/m = 0.2$  ( $\blacksquare$ ),  $1.0$  ( $\bullet$ ) and  $5.0$  ( $\blacktriangle$ ). The solid line is a plot of the approximate result derived in (37). **Inset:** Double logarithmic plot of the persistence probability of a Brownian particle in the overdamped limit. The solid line is power law fit to the data  $t^{-\theta}$ , with  $\theta = 0.482422$ . The analytical prediction for  $\theta$  in this limit is  $1/2$ .

diffusive, it suffices to consider the Langevin equation for a free particle for our present discussion.

The two-time correlation function for the velocities decays exponentially as  $\langle v(t_1)v(t_2) \rangle = \frac{k_B T}{m} e^{-|t_1 - t_2|/\tau_0}$  (with an initial condition  $\langle v^2(0) \rangle = \frac{k_B T}{m}$ ) and the position correlation,

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle &= \frac{k_B T}{m} \left[ \frac{2m}{\gamma} t_2 - \frac{m^2}{\gamma^2} \right. \\ &\quad \left. + \frac{m^2}{\gamma^2} \left( e^{-t_1/\tau_0} + e^{-t_2/\tau_0} - e^{-(t_1 - t_2)/\tau_0} \right) \right] \end{aligned} \quad (34)$$

with  $\tau_0 = m/\gamma$  and the assumption that  $t_1 > t_2$ . While the transformation of the correlator in (34) to a stationary process is nontrivial, we can still extract some information about the persistence probability in the limit of  $t \rightarrow 0$ . Keeping in mind this limit and that  $t_1 > t_2$ , a Taylor expansion of (34) yields,

$$\langle x(t_1)x(t_2) \rangle = \frac{t_1 t_2}{m} \left( 1 - \frac{1}{2} \frac{t_1}{\tau_0} \right) \quad (35)$$

The two-time correlation function  $\langle \bar{X}(t_1)\bar{X}(t_2) \rangle$  reads as,

$$\langle \bar{X}(t_1)\bar{X}(t_2) \rangle = \sqrt{\frac{1 - t_1/2\tau_0}{1 - t_2/2\tau_0}} \quad (36)$$

The transformation  $e^T = (1 - t/2\tau_0)^{-1}$  transforms (36) into a stationary process with an exponentially decaying

correlation function. The persistence probability  $p(t)$  in real time translates to,

$$p(t) \sim \frac{2}{\pi} \sin^{-1}[\sqrt{1 - t/2\tau_0}] \quad (37)$$

The numerical integration of (33) was done with an implicit integration scheme based on the Leap-Frog algorithm for different values of the ratio  $\gamma/m$  and the value of  $T$  was set to unity. The results of the simulations are presented in fig. 4 and compared to the approximate formula for  $p(t)$  in (37).

#### IV. CONCLUSION

In conclusion, we have investigated the persistence probability of a harmonically confined Brownian particle in the overdamped limit, with the potential relaxing to zero at long times. We consider two functional forms of the relaxation – an exponential and an algebraic relaxation. The simple model system presented in this article is analogous to a moving wall [30], with a “hard” wall re-

placed by a “soft” wall. The external confinement can be realized using a laser-trapping experiment, with the intensity of the laser decaying in time. When the confining potential relaxes exponentially, we observe that the dynamics of the Brownian particle at short and long times is purely diffusive and independent of the relaxation time scales. On the other hand, for an algebraic relaxation, the motion at long times is determined by the exponent of the relaxation. Using the two-time correlation function for the position of the Brownian particle, we construct the persistence probability of the Brownian particle in the two scenarios.

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